On the geometry of chiral dynamics. I

1970 J. Phys. A: Gen. Phys. 3 33

(http://iopscience.iop.org/0022-3689/3/1/005)

On the geometry of chiral dynamics: I

J. S. DOWKER
Department of Theoretical Physics, University of Manchester
MS. received 3rd September 1969

Abstract. The group-theoretical discussion of chiral dynamics is re-expressed in analytical terms appropriate to the geometry of group space. Closed, but transcendental, expressions are given for the meson Lagrangian for the cases of $G \otimes G$ in general and SU(2) $\otimes$ SU(2) in particular.

1. Introduction

The reproduction of the results of the partially conserved axial-vector current hypothesis and current algebra by phenomenological chiral-invariant Lagrangian methods has prompted a great deal of work in this field, and we refer to the review article by Gasiorowicz and Geffen (1969) for a detailed discussion and the relevant literature.

The mathematical basis of the theory seems to be that of non-linear realizations of the chiral group, and a general discussion can be found in the papers of Coleman, *et al.* (1969) and Callan *et al.* (1969). Quite generally the 0^- mesons are understood to form the coordinates on a manifold, in general curved, attached to each space–time point. Chiral invariant forms are then constructed according to the usual rules of tensor analysis. These forms are 'generally covariant' under analytic redefinitions of the meson fields, and it is easy to see that this means that only derivative couplings will emerge. Crudely speaking, the coordinates ξ^α are not tensors but the differentials $d\xi^\alpha$ are.

For the case of SU(2) $\otimes$ SU(2) Meetz (1969) has given an explicit geometrical interpretation of the non-linear realizations in terms of a 'curved isospin' space. A generalization to arbitrary chiral groups has been discussed by Isham (1969 a, b). The general ideas of this scheme are very attractive but the present author finds himself somewhat put off by the details of the formalism and wishes, in this paper, to give another discussion of this geometrical approach. No doubt all we shall be doing is using an 'old-fashioned' language that the author understands in place of one which he does not. However, the coordinate-free technique, while more modern, seems to be unnecessary, at least at this level.

2. The basic structure

We shall be specifically concerned with a chiral group K of the product form $G_r \otimes G_r$, where G_r is an r parameter, simple Lie group, for example SU(n), $n^2 = r+1$. When we wish to specify to which G_r group we refer we shall write $K = (G_r)_\mathrm{L} \otimes (G_r)_\mathrm{R}$, i.e. left and right groups. $(G_r)_\mathrm{L}$ and $(G_r)_\mathrm{R}$ are commutative and so are maximum invariant subgroups of K (Eisenhart 1933, p. 121).

We now introduce X_r the group manifold of G_r, i.e. the space coordinatized by the parameters ξ^α, η^α ..., labelling the elements of G_r, $\alpha = 1, 2 \ldots r$.

(For all definitions concerning spaces, coordinates, etc., we refer to the tensor analyst's Bible (Schouten 1954).)

The group K can now be interpreted as the product of the so-called first and second parameter groups of G_r, which are isomorphic to G_r itself. If we denote by ξ the group element (point) whose coordinates are ξ^α, then the group multiplication of G_r

$$'\xi = \eta\xi \tag{1}$$

gives point transformations $\xi \to {}'\xi$ which, for all η, form a group—the first parameter group of G_r—clearly isomorphic to G_r. Similarly the second parameter group is given by the family of transformations

$$\xi \to {}'\xi = \xi\eta. \tag{2}$$

In coordinates, (1) and (2) read

$${}'\xi^\alpha = F^\alpha[\xi, \eta] \tag{3}$$

and

$${}'\xi^\alpha = F^\alpha[\eta, \xi] \tag{4}$$

where the functions (functionals) F^α define the group combination law. Equations (3) and (4) provide, in fact, non-linear realizations of the abstract G_r.

We can thus represent K by the transformations

$$K: \qquad \xi = {}'\xi = \eta_R \xi \eta_L. \tag{5}$$

That subgroup of K defined by $\eta_L = \eta_R^{-1}$, where η^{-1} means the element inverse to η, is the adjoint group of G_r which is isomorphic to G_r if G_r is semi-simple. Let us denote the adjoint group by H.

The coordinates ξ^α (or the coordinate system (α)) on X are, here, quite general ones. An important role is played by the canonical coordinates. In terms of these, the transformations $\xi \to {}'\xi$ of the adjoint group take the form of linear homogeneous transformations, and we can now see the connection with the work of Coleman *et al.* (1969). For example, for chiral SU(2) the group H will just be the group of pure isospin transformations under which the pion field transforms linearly.

The quark fields (e.g. nucleons or 'spinors') are introduced in the same way as in general relativity, that is, locally (or, more accurately, pointwise) through the tangent space concept.

We now proceed to detail these ideas in analytical form.

3. The geometry of group space

The geometry of group space forms a topic of classical mathematics, the three basic mémoires being by Cartan (1927 a, b) and Schouten (1929). Discussions, from different viewpoints, can be found in the work of Eisenhart (1933) and Schouten (1954).

For this reason we do not want to develop the theory *ab initio* and shall just content ourselves with an outline, albeit long and standard.

As might be expected, one can develop the theory synthetically or analytically. We prefer the latter, as a rule, although it is often easier to see what is happening synthetically. However, we do not wish to burden ourselves, or the reader, with too many definitions and theorems. The synthetic approach will be found in Cartan (1927 a, b).

We begin by introducing the quantities

$$A^\alpha_a[\xi] = \left.\frac{\delta F^\alpha[\xi,\eta]}{\delta\eta^\beta}\right|_{\eta=\overset{\circ}{\eta}} \delta^\beta_a; \qquad A^\alpha_A[\xi] = \left.\frac{\delta F^\alpha[\eta,\xi]}{\delta\eta^\beta}\right|_{\eta=\overset{\circ}{\eta}} \delta^\beta_A$$

which appear in the classical theory of Lie groups. $\overset{\circ}{\eta}$ is some fixed point. These quantities arise when 'infinitesimal' transformations are considered. Thus, if we change the parameters η^α in (3) to $\eta^\alpha + \mathrm{d}\eta^\alpha$, then ${}'\xi^\alpha$ will be altered to ${}'\xi^\alpha + \mathrm{d}'\xi^\alpha$. We can also arrive at ${}'\xi + \mathrm{d}'\xi$ from ${}'\xi$ by an infinitesimal transformation, with parameters $\overset{\circ}{\eta}{}^\alpha + \overset{\circ}{\mathrm{d}}\eta^\alpha$, $\overset{\circ}{\eta}$ corresponding to the unit element of G_r or the 'origin'†. The relation between

† The origin will be represented by $\overset{\circ}{\eta}$, $\overset{\circ}{\xi}$ etc.

$d\eta^\alpha$ and $\underset{\circ}{d}\eta^\alpha$ is

$$d\eta^\alpha = A^\alpha_a[\eta]\delta^a_\beta \underset{\circ}{d}\eta^\beta.$$

We assume the boundary conditions

$$A^\alpha_a[\underset{\circ}{\eta}] = \delta^\alpha_a, \qquad A^\alpha_A[\underset{\circ}{\eta}] = \delta^\alpha_A. \tag{6}$$

We can look upon the quantities A^α_a and A_A as defining structures on the space X_r, specifically absolute parallelisms (for a history of this topic see Cartan 1930). In other words we look upon the $A^\alpha_a(A^\alpha_A)$ as giving at each point a set of r vectors labelled by the index $a(A)$. These vectors form a coordinate mesh, and parallelism is defined by saying that two general vectors at different points are parallel if they have the same coordinates with respect to the local coordinate systems formed by the A^α_a, A_A. The parallelism defined by the A^α_a is termed $(+)$ parallelism and that by A^α_A, $(-)$ parallelism. These parallelisms are absolute because there is only one vector at a given point parallel to a chosen vector at another point.

The quantities A^α_a and A^α_A define, in fact, two 'anholonomic' coordinate systems (a) and (A) (see Schouten 1954, p. 99). The anholonomic or local coordinates of a convariant vector v_α are

$$v_a = A^\alpha_a v_\alpha$$

and similarly for the (A) system. For a contravariant vector v^α we have

$$v^a = A^a_\alpha v^\alpha$$

where the A^a_α are the inverse or reciprocal set to A^α_a, i.e.

$$A^\alpha_a A^b_\alpha = \delta^b_a, \qquad A^\alpha_a A^a_\beta = \delta^\alpha_\beta.$$

In particular, the anholonomic components of $d\eta^\alpha$ will be

$$(d\eta)^a = A^a_\alpha \, d\eta^\alpha = \delta^a_\alpha \underset{\circ}{d}\eta^\alpha.$$

They are written $(d\eta)^a$ because $d\eta^a$ would imply they are the differentials of some functions $\eta^a(\eta^\alpha)$, which is not the case in general. This is the origin of the term anholonomic. It can easily be checked that two vectors $\underset{1}{v}^\alpha dt$ and $\underset{2}{v}^\alpha dt$ at $\underset{1}{\eta}$ and $\underset{2}{\eta}$ respectively are $(+)$ parallel if

$$(\underset{1}{\eta} + \underset{1}{v}\, dt)\underset{1}{\eta}^{-1} = (\underset{2}{\eta} + \underset{2}{v}\, dt)\underset{2}{\eta}^{-1} \tag{7}$$

and $(-)$ parallel if

$$\underset{1}{\eta}^{-1}(\underset{1}{\eta} + \underset{1}{v}\, dt) = \underset{2}{\eta}^{-1}(\underset{2}{\eta} + \underset{2}{v}\, dt). \tag{8}$$

Equations (7) and (8) are convariant under transformations of K, (5), i.e. equations (7) and (8) with all quantities replaced by primed quantities are valid if the relation between primed and unprimed is that given by (5). This means that two (infinitesimally close) parallel vectors are transformed by (5) into two other such vectors. In other words, the geometry around the point $'\xi$ is the same as that around ξ, the geometry being that of $(+)$ and $(-)$ parallelism. The group K is therefore a group of 'affine motions', or 'isometries'.

More particularly, it is simple to show that a vector and its transform† by the first (second) parameter group are $(-)$ $((+))$ parallel. It is clear from the definitions of the

† By transform we mean here that the transformed vector is obtained by joining the points obtained by the point transformation from the end points of the original vector, i.e. it is the 'dragged along' vector.

parallelisms that the vectors $A^\alpha_a[\eta]$ and $A_A[\eta]$ form vector fields which are (+) and (−) parallel respectively, and so are absolutely invariant if dragged along by point transformations belonging to the second and first parameter groups respectively.

Analytically the one-parameter subgroups of the first and second parameter groups are the trajectories obtained by integrating the equations

$$\frac{d\xi^\alpha}{dt} = e^a A^\alpha_a[\xi], \qquad e^a = \text{constants} \tag{9}$$

and

$$\frac{d\xi^\alpha}{dt} = e^A A^\alpha_A[\xi]. \tag{10}$$

We can then say that the Lie derivatives (Schouten 1954, p. 102) of A^α and A^α_A with respect to A^α_A and A^α_a respectively are zero:

$$\underset{a}{\pounds} A^\alpha_A = 0 = \underset{A}{\pounds} A^\alpha_a.$$

Since the relation between the components of two infinitesimally close parallel vectors will be a linear one, we can introduce linear connections for the two types of parallelism. Since the A^α_a and A^α_A are auto-parallel fields, we have

$$\overset{+}{\nabla}_\mu A^\alpha_a = 0 = \overset{-}{\nabla}_\mu A^\alpha_A$$

hence

$$\overset{+}{\Gamma}{}^\alpha_{\gamma\beta} = A^\alpha_a \partial_\gamma A^a_\beta, \qquad \overset{-}{\Gamma}{}^\alpha_{\gamma\beta} = A^\alpha_A \partial_\gamma A^A_\beta.$$

The curvatures corresponding to these two connections vanish, as follows by explicit calculations and also from the fact that the parallelism is absolute, i.e. the connections can be integrated (taking a vector around any closed curve by parallel displacement yields no change).

In general, the connections are not symmetric

$$\overset{\pm}{S}{}_{\gamma\beta}^{\cdot\cdot\alpha} = \overset{\pm}{\Gamma}{}^\alpha_{[\gamma\beta]} \neq 0$$

and the space is said to possess torsion.

The geodesics or paths in X_r are defined to be autoparallel curves and therefore have the equations

$$\frac{d^2\eta^\alpha}{dt^2} + \overset{\pm}{\Gamma}{}^\alpha_{\gamma\beta} \frac{d\eta^\gamma}{dt}\frac{d\eta^\beta}{dt} = 0 \tag{11}$$

if an affine parameter t is used. It can be shown synthetically that the (+) geodesics coincide with the (−) geodesics, so that from (11)

$$\overset{+}{\Gamma}{}^\alpha_{(\gamma\beta)} = \overset{-}{\Gamma}{}^\alpha_{(\gamma\beta)} = \Gamma^\alpha_{\gamma\beta}$$

which defines the symmetric connection $\Gamma^\alpha{}_\beta$, the so-called (0) connection. This connection is not integrable. Its curvature tensor $R_{\dot{\delta}\dot{\gamma}\beta}{}^\alpha$ does not vanish.

It is clear from (9) and (10) that the trajectories of the first parameter group are (−) auto-parallel and those of the second parameter group are (+) auto-parallel and therefore both coincide with the paths (11) determined by the (0) connection.

Other important relations that can be proved are

$$\overset{+}{\Gamma}{}^\alpha_{\gamma\beta} = \overset{-}{\Gamma}{}^\alpha_{\beta\gamma}$$

and so

$$\overset{+}{S}_{\gamma\beta}^{\cdot\cdot\alpha} = -\overset{-}{S}_{\gamma\beta}^{\cdot\cdot\alpha}.$$

Also we can show that dragging along over a ($\pm$) parallel infinitesimal field is the same as a ($\mp$) parallel displacement over the same field, i.e.

$$\underset{a}{\pounds}\Phi = A_a^\alpha \overset{-}{\nabla}_\alpha \Phi, \qquad \underset{B}{\pounds}\Phi = A_B^\alpha \overset{+}{\nabla}_\alpha \Phi.$$

For convenience of having the correct factors we define the quantity $c_{\gamma\beta}^{\cdot\cdot\alpha}$ by

$$c_{\gamma\beta}^{\cdot\cdot\alpha} = -2\overset{+}{S}_{\gamma\beta}^{\cdot\cdot\alpha}$$

and in terms of which the curvature of the (0) connection is given by

$$R_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha} = -\tfrac{1}{4}c_{\delta\gamma}^{\cdot\cdot\epsilon}\, c_{\epsilon\beta}^{\cdot\cdot\alpha}.$$

Further, it can be shown that $c_{\gamma\beta}^{\cdot\cdot\alpha}$ is (+), (−) and (0) constant

$$\nabla_\delta c_{\gamma\beta}^{\cdot\cdot\alpha} = 0 \tag{12}$$

and so we have the important result that the curvature $R_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha}$ is covariant constant:

$$\nabla_\epsilon R_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha} = 0. \tag{13}$$

The anholonomic components of the antisymmetric part of the ($\pm$) connections are just the structure constants of the group G_r. Thus we have for these components

$$c_{cb}^{a} = -c_{bc}^{a} = c_{\gamma\beta}^{\cdot\cdot\alpha} A_\alpha^a A_b^\beta A_c^\gamma$$

(remembering that $c_{\gamma\beta}^{\cdot\cdot\alpha}$ is a tensor in contrast to the symmetric part) and from the condition $\overset{+}{\nabla} A_a^\alpha = 0$ there follows Maurer's equation

$$\partial_{[\gamma} A_{\beta]}^a = -\tfrac{1}{2} c_{cb}^{a} A_\gamma^c A_\beta^b.$$

It can be proved, in the standard manner, that the c_{cb}^{a} are constants. If we wish we can use the (A) system of anholonomic coordinates and, in fact, it is convenient to choose the quantities A_a^α and A_A^α so that they coincide at the origin

$$A_a^\alpha[\underset{\circ}{\eta}] = A_A^\alpha[\underset{\circ}{\eta}].$$

Then we have

$$c_{CB}^{A} = \delta_a^A \delta_B^b \delta_C^c c_{cb}^{a}.$$

To show that the c_{bc}^{a} are the structure constants occurring in the equation for the commutator of the generators, we consider infinitesimally small transformations of the parameter groups. We already have these in (9) and (10). We define the infinitesimal operators ∂_a and ∂_A so that

$$\mathrm{d}\xi^\alpha = te^a \partial_a \xi^\alpha, \qquad \mathrm{d}\xi^\alpha = te^A \partial_A \xi^\alpha.$$

Thus for the commutators we have the structural formulae

$$[\partial_c, \partial_b] = c_{cb}^{\cdot\cdot a} \partial_a, \qquad \partial_a = A_a^\alpha \partial_\alpha$$

and

$$[\partial_C, \partial_B] = -c_{CB}^{\cdot\cdot\cdot A}\partial_A, \qquad \partial_A = A_A^\alpha \partial_\alpha.$$

Integrating equations (9) and (10) yields the finite forms of the parameter groups

$$\eta^\alpha = \exp(te^a\partial_a)\overset{\circ}{\eta}^\alpha, \qquad \eta^\alpha = \exp(te^A\partial_A)\overset{\circ}{\eta}^\alpha$$

where we have chosen the origin as initial point. These equations give the relation between the canonical parameters te^a and the η^α, i.e. the coordinates η^α and $\eta^a = te^a$ refer to the same point in group space. In terms of η^a the paths become simply straight lines and the corresponding transformations can be called translations.

As explained by Schouten (1929), a different type of index must be employed for the canonical coordinate system. We shall use latin letters from the middle of the alphabet, i, j etc. Thus we define the canonical coordinates by

$$\eta^i = \delta_a^i te^a = \delta_a^i \eta^a \dagger$$

and we shall have the intermediate quantities A_i^a, A_i^α which enable us to pass from one system to another. We assume that all the coordinate systems (α), (a), (A) and (i) coincide at the origin, i.e.

$$\delta_i^\alpha = A_i^\alpha[\overset{\circ}{\eta}], \qquad A_a^\alpha[\overset{\circ}{\eta}] = \delta_a^\alpha, \qquad A_a^i[\overset{\circ}{\eta}] = \delta_a^i.$$

Canonical coordinates are characterized by the condition

$$\eta^i A_i^a[\eta] = \eta^i \delta_i^a$$

which enables us to calculate the quantity A_i^a. It is found that‡

$$\begin{aligned} A_i^a &= \delta_i^b\left(\delta_b^a - \frac{1}{2!}U_b^a + \frac{1}{3!}U_b^c U_c^a - \ldots\right) \\ &= \delta_i^b\left(\frac{1-e^{-U}}{U}\right)_b^a \\ A_a^i &= \delta_b^i\left(\frac{U}{1-e^{-U}}\right)_a^b = \delta_b^i(1-\tfrac{1}{2}U+\tfrac{1}{12}U^2-\ldots)_a^b \end{aligned} \tag{14}$$

with

$$U_b^a = \eta^c c_{cb}^{\;\;a}.$$

Similarly it is found for A_i^A

$$A_i^A = \delta_i^b \delta_a^A \left(\frac{e^U-1}{U}\right)_b^a \;. \tag{15}$$

The importance of canonical coordinates lies in the fact that in terms of them the transformations of the adjoint group become linear and homogeneous, as mentioned before. The adjoint group plays an important role in the classification of the structure of Lie groups. We have defined the adjoint group as the group of inner automorphisms

$$\xi \to {}'\xi = \eta\xi\eta^{-1}. \tag{16}$$

† For η itself we can use i or a, similarly $d\eta^i$ or $d\eta^a$, but we must remember not to confuse this last with $(d\eta)^a = A_\alpha^a d\eta^\alpha$.

‡ This construction is due to Schur (1891).

If canonical coordinates are used for ξ and $'\xi$, we find the representation of (16) as

$$'\xi^i = D^i_j[\eta]\xi^j \tag{17}$$

where the matrices, for each element η of the group G_r, form the adjoint representation of G_r. In terms of canonical parameters for η we have

$$D^i_j[\eta] = \delta^i_a\delta^b_j[\exp(-U)]^a_b \equiv \delta^i_a\delta^b_j D^a_b \tag{18}$$

where

$$U^a_b = \eta^c c^a_{cb}$$

or, in terms of infinitesimal operators Ξ_a,

$$'\xi^i = \exp\{\eta^a\Xi_a(\xi)\}\xi^i$$

where

$$\Xi_a(\xi) = \xi^c c^b_{ca}\frac{\partial}{\partial\xi^b}.$$

These operators have the commutator

$$[\Xi_a, \Xi_b] = c^c_{ab}\Xi_c.$$

For a general matrix representation the relation between the generators J_b, as the word is commonly used†, and the infinitesimal operators or Lie symbols X_b is

$$X_b = \mathrm{i}\varphi^i(J_b)^j_i\frac{\partial}{\partial\varphi^j}$$

so that the transformation in representation space reads

$$'\varphi^j = \exp(\lambda^b X_b)\varphi^j = [\exp(\mathrm{i}\lambda^b J_b)]^j_i\varphi^i.$$

Hence for the adjoint representation the generators are the matrices whose components are the structure constants, as is well known.

In terms of an exponential mapping, (16) can be written

$$\exp(-\eta^a X_a)\exp(\xi^a X_a)\exp(\eta^a X_a) = \exp('\xi^a X_a).$$

We are here considering the group elements as operators on some unspecified manifold, although this is not necessary.

Equations (17) and (18) can be derived directly in terms of canonical coordinates in the case where η corresponds to an infinitesimal operation. Thus we perform a transformation of the first parameter group with parameters $\underset{\circ}{\mathrm{d}}\eta^a$ and follow it with a transformation of the second group with parameters $-{}_\circ\eta^A = -\delta^A_a\underset{\circ}{\mathrm{d}}\eta^a$. For the total change in ξ we find

$$\mathrm{d}\xi^i = (A^i_a[\xi] - A^i_A[\xi]\delta^A_a)\underset{\circ}{\mathrm{d}}\eta^a$$

and, using the expressions (14) and (15), this reduces to‡

$$\mathrm{d}\xi^i = U^b_a(\xi)\,\underset{\circ}{\mathrm{d}}\eta^a\delta^i_b = -\underset{\circ}{\mathrm{d}}\eta^c c^b_{ca}\delta^i_b\delta^a_j\xi^j$$

which is just (17) in infinitesimal form.

† We make J_b Hermitian for unitary groups.

‡ cf. Kowalewski (1931) pp. 170–87.

For future reference we give the change in ξ for the infinitesimal transformation

$$\xi \rightarrow \eta\xi\eta$$

which represents a 'pure chiral' transformation. We find

$$d\xi^i = [U(\xi)\coth\{\tfrac{1}{2}U(\xi)\}]^b_a\, d\overset{\circ}{\eta}{}^a \delta^i_b. \tag{19}$$

The adjoint group can be looked upon in various guises. According to (16) it is an auto-morphism of the group G_r, and, as such, clearly leaves the structure constants numerically invariant. In symbols, if

$$e^a \partial_a = {}'e^{a\prime}\partial_a$$

defines $'\partial_a$, where e^a and $'e^a$ are related by an adjoint transformation,

$$'e^a = (e^{-U})^a_b e^b$$

we have

$$[' \partial_a, '\partial_b] = c^c_{ab}\, '\partial_c.$$

We say that quantities like ∂_a with a lower index transform according to the co-adjoint representation. The numerical invariance of c^c_{ab} can be easily checked.

The adjoint group transforms geodesics through $\overset{\circ}{\eta}$ into geodesics through $\overset{\circ}{\eta}$ and, since the canonical coordinates of a point depend essentially on the geodesic on which it sits, i.e. on the tangent vector at the origin, we can see why they are transformed linearly. We can thus interpret the e^a as Cartesian coordinates of a flat affine space attached to the point $\overset{\circ}{\eta}$, i.e. a tangent space of $\overset{\circ}{\eta}$. The adjoint group is then a group of linear transformations of this tangent space which leave $\overset{\circ}{\eta}$ unchanged.

4. Group space as a Riemannian space

Since the c^c_{ab} are invariant under the adjoint group so are all the quantities that can be constructed from them. A particularly important set is given by the following traces of multiples of the matrices C_a:

$$\begin{aligned}
|C_a|^c_b &\equiv c^c_{ab} \\
g_a &= \mathrm{Tr}\, C_a \\
g_{ab} &= \alpha\, \mathrm{Tr}(C_a C_b), \qquad \alpha = \text{constant} \\
g_{abc} &= \mathrm{Tr}(C_a C_b C_c) \\
&\vdots
\end{aligned}$$

and Cartan has shown that the group G_r is semi-simple if the matrix whose elements are g_{ab} is non-singular (or has rank r). In this case we can think of the g_{ab} as a metric in the adjoint representation space, i.e. the tangent space. The adjoint and co-adjoint representations are equivalent and we pass from one to the other by raising and lowering indices using the metric g_{ab} and its inverse g^{ab}.

We now use g_{ab} to introduce a metric into group space. Thus we define $g_{\alpha\beta}$ by

$$g_{\alpha\beta} = A^a_\alpha A^b_\beta g_{ab} \tag{20}$$

so that the g_{ab} are the anholonomic components of $g_{\alpha\beta}$. In terms of the $c_{\dot{\alpha}\dot{\beta}}{}^{\gamma}$ we have

$$g_{\alpha\beta} = \alpha \mathrm{Tr}(C_\alpha C_\beta)$$

and so contraction of the curvature

$$R_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha}, = -\tfrac{1}{4}(C_\gamma C_\beta)_\delta^\alpha$$

yields

$$R_{\gamma\beta} \equiv R_{\alpha\gamma\beta}^{\cdot\cdot\cdot\alpha} = -\alpha\tfrac{1}{4}g_{\gamma\beta}.$$

Similar constructions in terms of the (A) anholonomic system are possible and we find the equality

$$g_{\alpha\beta} = A_\alpha^A A_\beta^B g_{AB}, \qquad g_{AB} = \delta_A^a \delta_B^b g_{ab}. \tag{21}$$

But we already have that the curvature is (0) covariant constant, and so therefore is $g_{\alpha\beta}$

$$\nabla_\gamma g_{\alpha\beta} = 0.$$

This implies that the space is Riemannian. This particular construction is only possible for semi-simple groups. It *is* possible to define metric structures and Riemannian geometry for the group space of any group, but the Riemannian geodesics do not, in general, coincide with the trajectories of the group, which is somewhat displeasing. We shall restrict ourselves to semi-simple groups, particularly to simple compact ones.

We emphasize that the Riemannian geometry is of a very special type, being such that the curvature is covariant constant, i.e. conserved under parallel transport. Spaces with this property are termed 'symmetric' (Schouten 1954, p. 163), for a reason we shall encounter shortly. They have been studied both from the group-theoretical and purely geometrical aspects, historic papers being by Cartan (1927 a, b, 1929).

For compact simple groups like SU(n) we can always make, by an appropriate choice of coordinates in adjoint space, $\mathrm{Tr}(C_a C_b)$ proportional to δ_{ab}, and so we shall choose g_{ab} to equal δ_{ab}. The matrix C_a is antisymmetric.

5. Motions in group space: chiral invariant structures

We have mentioned these earlier and now want to discuss them analytically, confining ourselves to the case of semi-simple groups, i.e. to motions in a Riemannian space. By definition, a point transformation $\xi \to {}'\xi$ is a motion if the following condition is satisfied:

$$g_{\alpha\beta}(\xi)\, \mathrm{d}\xi^\alpha\, \mathrm{d}\xi^\beta = g_{\alpha\beta}({}'\xi)\, \mathrm{d}'\xi^\alpha\, \mathrm{d}'\xi^\beta. \tag{22}$$

This means simply that the geometry in the region around $'\xi$ is identical with that around ξ. Formally, (22) implies that the Lie derivative† of $g_{\alpha\beta}$ vanishes over the transformation $\xi \to {}'\xi$ and, if this latter belongs to K (= (1st parameter group) $\otimes$ (2nd parameter group)), $\pounds g_{\alpha\beta}$ *does* vanish as follows from (20), (21), and the vanishing of the Lie derivatives of the A_α^a and A_α^A (or, equivalently, of the $c_{\alpha\beta}^{\cdot\cdot\gamma}$).

We can now see how to obtain chiral invariant structures. Expression (22) is one of the simplest. All we do is associate $\lambda^{-1}\xi^\alpha$ with the appropriate meson field (e.g. pion) and note that the space–time derivative $\partial\xi^\alpha/\partial x^\mu$ transforms like $\mathrm{d}\xi^\alpha$ under coordinate transformations $\xi^\alpha \to \xi^{\alpha'}$, so that, with Meetz (1969) and Isham (1969), we may replace $\mathrm{d}\xi^\alpha$ by $\partial_\mu \xi^\alpha$ and obtain the kinematic part of the meson Lagrangian,

$$-\tfrac{1}{2}\lambda^{-2} g_{\alpha\beta}(\xi)\partial_\mu\xi^\alpha\partial^\mu\xi^\beta.$$

If we use canonical coordinates the explicit construction of the metric is easy. Thus from (14) and (15) we have

$$g_{ij} = A_i^a A_j^b \delta_{ab} = \delta_i^c \delta_j^d \left[\left\{\frac{1-\exp(-U)}{U}\right\}\left\{\frac{1-\exp(-\tilde{U})}{\tilde{U}}\right\}\right]_{cd}$$

$$\tilde{U} = -U$$

† See Schouten (1954, p. 346). There are various ways of describing the situation.

whence

$$g_{ij} = 2\delta_i^a\delta_j^b\left[\frac{\cosh U - 1}{U^2}\right]_{ab}$$
$$= \delta_i^a\delta_j^b[1 + \tfrac{1}{12}U^2 + \tfrac{1}{72}U^4 + \ldots]_{ab}. \tag{23}$$

We note that the expansion in terms of U^2 is just an expansion in the curvature by virtue of the relation

$$[U^2]_{da} = -4\xi^b\xi^c R_{dcba}.$$

Replacing ξ^α by λM^α, the meson field, the Lagrangian reads

$$\mathscr{L} = \left(\frac{\cosh U - 1}{U^2}\right)_{ab} \partial^\mu M^a \partial_\mu M^b \equiv \partial^\mu \tilde{M}\left(\frac{\cosh U - 1}{U^2}\right)\partial_\mu M$$
$$= \tfrac{1}{2}\partial^\mu M^a \partial_\mu M^a - \tfrac{1}{6}\lambda^2 R_{dcba}\partial^\mu M^d M^c M^b \partial_\mu M^a + \ldots . \tag{24}$$

If we allow for a difference in normalization this is identical with the expression of Callan *et al.* (1969),

$$\tfrac{1}{2}D_\mu\xi D^\mu\xi$$

$D_\mu\xi$ being a 'covariant derivative'.

The comparison of (24) with the series given by Isham (1969) shows that our λ is twice his. This is consistent with the paper of Callan *et al.* (1969).

Again for comparison we give the expression we find for the analogue of the f_{ab} quantities of Weinberg (1968) and generalized to arbitrary groups by Macfarlane *et al.* (to be published). In our notation this quantity should be denoted by F_a^j and is just the change in the meson field ξ^j for an infinitesimal purely chiral transformation with parameters $d\eta^b = \delta_a^b dt$. We have already computed this and equation (19) gives

$$F_a^j(\xi) = [U(\xi)\coth\{\tfrac{1}{2}U(\xi)\}]_a^b\delta_b^j. \tag{25}$$

Other chiral invariant quantities for ξ^α can be constructed simply by combining the concomitants of $c_{\dot\alpha\dot\beta}{}^\gamma$, for example $g_{\alpha\beta\gamma}$, with appropriate numbers of $d\xi^\alpha$ to form scalars in X_r. These will all be 'higher derivative' terms. The algebraic problems involved are just those of the linear adjoint group since we can always work in an anholonomic coordinate system. All the invariants with which we are familiar in 'ordinary' SU(n) theory, for example, can easily be transcribed into chiral invariant (differential) forms.

The particularly simple, i.e. familiar, algebra of SU(2) allows us to take the expressions (23) and (25) a little further in this case†. Now we have $c_{abc} = \epsilon_{abc}$ and U^2 reduces to $\xi \otimes \xi - \xi^2 1$, where $\xi^2 = \xi^a\xi^a \equiv \xi^a\xi_a$. Hence

$$U^4 = -\xi^2 U^2, \text{ etc.}$$

and we find for g_{ij} the form

$$g_{ij} = \delta_{ij}\left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2 + \frac{\xi_a\xi_b}{\xi^2}\delta_i^a\delta_j^b\left\{1 - \left(\frac{\sin\frac{1}{2}\xi}{\frac{1}{2}\xi}\right)^2\right\}$$

and for the F_a^j quantity

$$F_a^j = -\xi\cot(\tfrac{1}{2}\xi)\delta_a^j + \frac{2}{\xi^2}\{1 - \tfrac{1}{2}\xi\cot(\tfrac{1}{2}\xi)\}\xi_a\xi^j.$$

† The relevant algebra is detailed in Kowalewski (1931, p. 175 *et seq.*), where the case of SU(2) is discussed in detail relevant to the considerations of the present paper.

Comparing with Weinberg's equation (2.10) (Weinberg 1968), we see that we have

$$f = -\xi \cot(\tfrac{1}{2}\xi), \qquad g = \frac{2}{\xi^2}(1+\tfrac{1}{2}f).$$

It can be checked that these functions satisfy Weinberg's equation (2.11).

The group space of SU(2) is a three-dimensional space of constant (positive) curvature, i.e. a three-sphere with appropriate topology†. Its geometry has been studied intensively, forming, as it does, one of the two original non-Euclidean geometries. The other one, hyperbolic geometry, corresponds to a non-compact group. Cartan, particularly, has discussed the significance of group space in relation to the classification of geometries achieved by Klein in his 'Erlanger Program'. The whole subject is an extremely fascinating one, but is not mathematically fashionable‡.

We refer to the papers of Meetz (1969) and Isham (1969) for discussions on the geometry of SU(2) from the chiral viewpoint.

It should be clear where the term 'chiral' comes in but, in order to maintain the pedagogical nature of this paper, we shall briefly deal with this question.

It can be proved that the biggest connected group of isometries is, in fact, the group K of transformations

$$\xi \to {}'\xi = \eta_L \xi \eta_R \tag{26}$$

i.e. the direct product of left and right translations. However, there are other isometries. One such is

$$\xi \to {}'\xi = \xi^{-1}$$

termed 'reflection through the origin'. More generally we have 'reflection through the point ζ':

$$\xi \to \zeta\xi^{-1}\zeta. \tag{27}$$

The set of such transformations forms not a group but a family. They do form a group when combined with (26). This disconnected group is not, in general, the biggest group of isometries. Cartan (1927 b) has shown, for semi-simple groups, that there are exceptional cases where the isometry group consists of four or twelve separate parts. We shall not be concerned with these (interesting) pathological cases.

In terms of canonical coordinates, with respect to the point in question, ζ, the transformation (27) amounts to reversing the sign of the coordinates, i.e. it sends a point along the geodesic connecting it with ζ to an equal distance on the opposite side—hence 'reflection'. Analytically it can be checked that, again in canonical coordinates,

$$\overset{+}{\Gamma}{}^i_{kj}(\xi) = -\overset{+}{\Gamma}{}^i_{kj}(-\xi)$$

so that

$$\Gamma^i_{kj}(\xi) = -\Gamma^i_{kj}(-\xi)$$

and also

$$g_{ij}(\xi) = g_{ij}(-\xi)$$

and geodesics go over into geodesics. Further, if two $(+)$ parallel vectors are reflected they become $(-)$ parallel. In other words, the first parameter group changes place with the second one under reflection. This can, of course, be shown directly. If we identify the group and its first parameter group, then the group of inverse transformations is to be identified with the second parameter group (see Kowalewski 1931, p. 253). If for 'reflection' we read 'parity transformation', then the origin of the word 'chiral' is clear if we remember that the mesons are pseudo-scalar ones.

† We pass by all such considerations of connectedness.

‡ It appears, as a detail, in the modern theory of Lie groups (e.g. Helgason 1968).

Spaces for which reflections through all points are isometries are called 'symmetric' spaces. The necessary and sufficient condition for this is that the curvature should be covariant constant (e.g. Schouten 1954, Cartan 1927 a). Group space is thus a very special sort of space, one with a very special structure.

6. Conclusion

We have re-expressed the rather group-theoretic treatment of chiral dynamics, as given, for example, by Coleman *et al.* (1969) and Isham (1969), in rather more analytic language, which we find more understandable, particularly in view of the strong geometrical flavour of the terminology. Such a re-expression may have more than personal value in that it allows us to examine the historical development of the mathematics with an eye on its possible physical relevance to the chiral problem.

One possibility with which the author has been toying is that of translating the bootstrapping of internal symmetries (e.g. Leutwyler and Sudarshan 1967) into geometrical terms. In previous work the author was struck by the similarities of some of the equations that resulted from self-consistency to those occurring in Riemannian geometry and the notation used (Dowker 1964) was adjusted accordingly. It is perhaps significant that both theories, chiral dynamics and self-consistent symmetries, place restrictions on the type of diagrams employed, or needed.

The present discussion is incomplete as we still have the 'other fields' problem, i.e. the introduction of quarks and other particles. This will be dealt with at another time.

References

Callan, C. G., Coleman, S., Wess, J., and Zumino, B., 1969, *Phys. Rev.*, **177**, 2247–50.

Cartan, E., 1927 a, *J. Math. Pures Appl.*, **6**, 1–119.

—— 1927 b, *Annali Mat. Pura Appl.*, **4**, 209–56.

—— 1930, *Math. Ann.*, **102**, 698–706.

—— 1929, *J. Math. Pures Appl.*, **8**, 1–33.

Coleman, S., Wess, J., and Zumino, B., 1969, *Phys. Rev.*, **177**, 2239–47.

Dowker, J. S., 1964, *Nuovo Cim.*, **34**, 773–5.

Eisenhart, L. P., 1933, *Continuous Groups of Transformations* (Princeton, N.J.: Princeton University Press).

Gasiorowicz, S., and Geffen, D. A., 1969, *Rev. Mod. Phys.*, **41**, 531–73.

Helgason, S., 1968, *Battelle Rencontres* (New York: W. A. Benjamin).

Isham, C. J., 1969, *Nuovo Cim.*, **61A**, 188–202, 729–36.

Kowalewski, G., 1931, *Einführung in die Theorie der Kontinuerlichen Gruppen* (Leipsig: Akad. Verlag).

Leutwyler, H., and Sudarshan, E. C. G., 1967, *Phys. Rev.*, **156**, 1637–43.

Meetz, K., 1969, *J. Math. Phys.*, **10**, 589–93.

Schouten, J. A. 1929, *Math. Ann.*, **102**, 244–72.

—— 1954, *Ricci-Calculus* (Berlin: Springer-Verlag).

Schur, F., 1891, *Math. Ann.*, **38**, 263–86.

Weinberg, S., 1968, *Phys. Rev.*, **166**, 1568–77.